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Computer-Aided Analysis of Arbitrarily Shaped Coaxial Discontinuities

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Abstract —This paper proposes a method of analyzing a coaxial discontinuity arbitrarily shaped in two dimensions (radial and longitudinal) but maintaining its axial symmetry. It is shown that under such assumptions the equations to be solved correspond to the equations describing an equivalent planar circuit filled with anonuniform medium. These equations are solved by a version of the finite-difference time-domain method. The method produces a universal computer algorithm capable of solving a wide range of practical problems with no analytical preprocessing. The examples presented show that the method can be effectively used in engineering applications.

I. Introduction

Analysis of coaxial discontinuities was a hot issue some 20 or more years ago, when the foundations of precise coaxial techniques were being developed. At that time most simple types of discontinuities such as steps in inner or outer conductors [1] and open end capacitances [2] were analyzed and characterized in the form of formulas, curves, and tables [4].

Today the coaxial technique is still of great importance. Its foundations are well known, but a designer wishing to introduce an unconventional discontinuity is faced with a formidable problem. He can use the old curves and tables but they concern in most cases isolated discontinuities, and fail when there are several of them close enough to interfere with each other. Another possibility is to apply numerical techniques. A straightforward approach is to expand the fields in series of waveguide modes of axial symmetry and try to satisfy the boundary conditions. The variational method may be supportive in the effort [3] but since these series converge slowly, the procedure is complicated even for simple discontinuities and becomes impractical for more complex ones.

This paper proposes a method for analyzing a coaxial discontinuity arbitrarily shaped in two dimensions (radial and longitudinal) but maintaining its axial symmetry. It is shown that under such assumptions the equations to be solved can be transformed to a form identical with the equations describing an equivalent planar circuit filled with an inhomogeneous medium. These equations are solved by a version of the finite-difference time-domain (FD-TD) method. The method is intended to produce a universal computer algorithm capable of solving a wide range of practical problems with no analytical preprocessing and with the user's involvement reduced to defining the shape of the analyzed discontinuity.

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II. Planar Equivalent Circuit of a Coaxial Discontinuity

Let us assume an example of a circuit of axial symmetry which is arbitrarily shaped in two dimensions (axial z and radial r) as shown in the cross section in Fig. 1. We assume that the circuit is excited by a TEM mode of the coaxial line entering the circuit from the left. Because of the axial symmetry of both the input wave and the boundary conditions, the wave remains axially symmetrical throughout the circuit and can be described in any point of the circuit by a magnetic field vector with one component:

$$\boldsymbol{H} = \boldsymbol{a}_\phi \psi_h(r,z) e^{j\omega t} \tag{1}$$

where $\boldsymbol{a}_\phi$ is the unit vector in the azimuthal direction in the cylindrical coordinate system. The $\boldsymbol{E}$ field may be calculated from the equation

$$\boldsymbol{E} = \frac{1}{j\omega\epsilon} \nabla \times \boldsymbol{H}. \tag{2}$$

Thus the problem of field analysis in the circuit under study is reduced to a two-dimensional problem with $\psi_h(r,z)$ being an unknown function.

To solve the problem let us introduce two auxiliary functions:

$$\boldsymbol{J} = \boldsymbol{a}_\phi \times \boldsymbol{E} \tag{3}$$

and

$$V = -rH_\phi. \tag{4}$$

Taking into account (1), (2), (3), and (4), we obtain

$$\boldsymbol{J} = \frac{1}{j\omega\epsilon} \boldsymbol{a}_\phi \times (\nabla \times \boldsymbol{H}) = \frac{j}{\omega\epsilon r}\left[\boldsymbol{a}_r \frac{\partial V}{\partial r} + \boldsymbol{a}_z \frac{\partial V}{\partial z}\right]. \tag{5}$$

The Maxwell equation

$$\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H} \tag{6}$$

used with (1), (3), and (4) gives

$$\frac{\partial J_z}{\partial z} + \frac{\partial J_r}{\partial r} = -\frac{j\omega\mu V}{r}. \tag{7}$$

Equations (5) and (7) can be rewritten in a form more

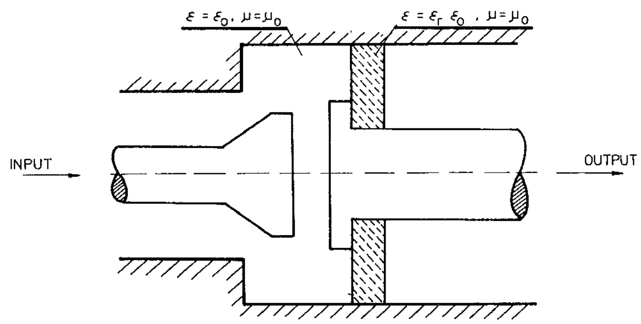

Fig. 1. A coaxial discontinuity.

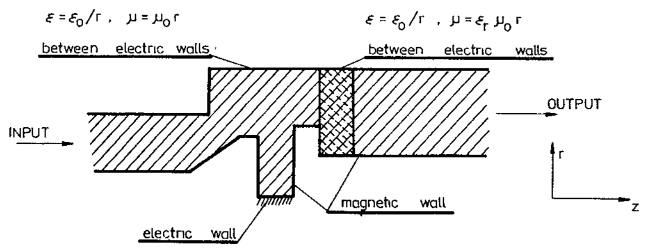

Fig. 2. Equivalent planar circuit of the discontinuity of Fig. 1.

convenient for further discussion:

$$\nabla_s V = -L_s \frac{\partial \boldsymbol{J}}{\partial t} \tag{8}$$

$$\nabla_s \cdot \boldsymbol{J} = -C_s \frac{\partial V}{\partial t} \tag{9}$$

where $L_s = r\epsilon$, $C_s = \mu / r$, and ∇_s is the two-dimensional nabla operator in rectangular coordinates r and z.

Equations (8) and (9) are identical with those for a planar two-dimensional circuit [6] filled with an inhomogeneous medium. Thus, instead of analyzing the circuit of Fig. 1, we may analyze its planar equivalence in Fig. 2.

III. Analysis of the Equivalent Planar Circuit

There are several methods of planar circuit analysis, but if we assume that the method we are looking for must allow effective calculations of the frequency-dependent characteristics of an inhomogeneously filled circuit with no analytical preprocessing, the choice is limited to two: the FD-TD method [5]–[7], [11] and the transmission-line matrix (TLM) method [8], [9]. Of these two methods the FD-TD method was found to be more effective in application to circuits such as that of Fig. 2 and will be discussed later.

To analyze the circuit of Fig. 2, we follow the approach described in [6], [7], and [12]. The circuit is divided into a set of meshes (Fig. 3); these are basically square of size a but can be modified to match the boundary shape. If the coordinates of the node (lying in the middle of the mesh) in the kth row and the lth column are denoted by z_l and r_k, replacing the differentials in (8) and (9) by finite differences Δt and a yields

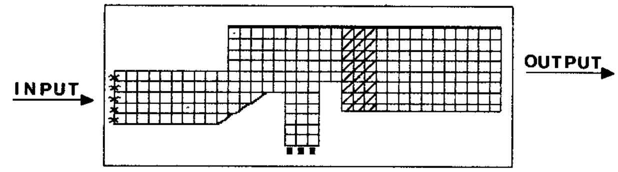

Fig. 3. Equivalent planar circuit of Fig. 2 as a set of meshes prepared for FD-TD calculations.

$$J_z\left(z_l + \frac{a}{2}, r_k, t_0 + \frac{\Delta t}{2}\right) = J_z\left(z_l + \frac{a}{2}, r_k, t_0 - \frac{\Delta t}{2}\right) - \left(V(z_l + a, r_k, t_0) - V(z_l, r_k, t_0)\right)\frac{\Delta t}{aL_s(r_k) f_1(l,k)} \tag{10}$$

$$J_r\left(z_l, r_k + \frac{a}{2}, t_0 + \frac{\Delta t}{2}\right) = J_r\left(z_l, r_k + \frac{a}{2}, t_0 - \frac{\Delta t}{2}\right) - \left(V(z_l, r_k + a, t_0) - V(z_l, r_k, t_0)\right) \cdot \frac{\Delta t}{aL_s(r_k + a/2) f_2(l,k)} \tag{11}$$

$$V(z_l, r_k, t_0 + \Delta t) = V(z_l, r_k, t_0) - \left(J_z\left(z_l + \frac{a}{2}, r_k, t_0 + \frac{\Delta t}{2}\right) - J_z\left(z_l - \frac{a}{2}, r_k, t_0 + \frac{\Delta t}{2}\right) + J_r\left(z_l, r_k + \frac{a}{2}, t_0 + \frac{\Delta t}{2}\right) - J_r\left(z_l, r_k - \frac{a}{2}, t_0 + \frac{\Delta t}{2}\right)\right) \cdot \frac{\Delta t}{aC_s(r_k) f_3(l,k)} \tag{12}$$

where $f_1(l,k)$, $f_2(l,k)$, and $f_3(l,k)$ are mesh shape functions which are equal to unity for all the meshes inside the circuit but can adopt different values (calculated by a boundary matching procedure) for meshes lying at the circuit's boundary.

The analysis is conducted by using (10), (11), and (12) to simulate the wave propagation in the circuit excited by a matched pulse source and terminated by a matched load. The Fourier transform is used to obtain frequency-dependent S matrix parameters.

The equivalent planar circuit describing a coaxial discontinuity has to be calculated with relatively high accuracy. To ensure that accuracy without boosting the computing time, we must consider some aspects of the FD-TD method.

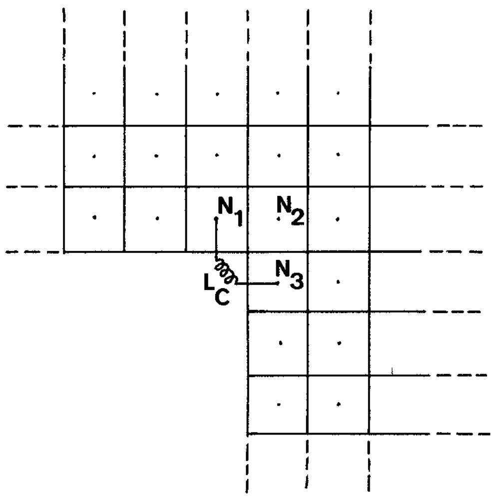

Fig. 4. A corner of an equivalent circuit with the correcting inductance.

A. Boundary Matching Procedure

Theoretically, any accuracy requirements of an FD method can be met by reducing the mesh size. However, when the mesh size decreases n times, the time of computing rises n^3 times because the number of meshes rises n^2 times. The step in time has to be cut by the factor of n to ensure algorithm stability [6]. This underlines the importance of procedures allowing better boundary approximation without decreasing the mesh size. Such a procedure was introduced in [6]. It will be used here in a form modified to include the circuit's inhomogeneous filling and supplemented by the right angle correction as explained below.

Let us assume a fragment of a planar circuit including a right angle corner (Fig. 4), with the nodes close to the corner denoted by N_1, N_2, N_3. In this fragment of the circuit the current flowing around the corner tends to concentrate near it while the FD algorithm assumes that it passes through the node N_2. That is why in the FD-TD calculations, when a corner such as that of Fig. 4 is considered, the additional inductance L_c is added to connect the nodes N_1 and N_3. After numerical experiments this value was chosen to be

$$L_c = 4(L_{12} + L_{23}) \tag{13}$$

where L_{12} and L_{23} are inductances between the corresponding nodes assumed normally in the FD-TD algorithm.

B. Modeling of Matched Loads and Sources

When compensated coaxial discontinuities are analyzed, the associated levels of the reflection coefficient are very low. That is why the matched loads and sources have to be modeled in the algorithm with great accuracy. Such accuracy can be obtained with the procedure described in [12]. Fig. 5 presents the results of calculations of $|S_{11}|$ versus frequency (characterized by the ratio of mesh size to wavelength a/λ) for a uniform line having a length of 20 meshes. It is seen that when the procedure of [12] is used, the errors of matching drop to negligible values even for relatively high a/λ ratios.

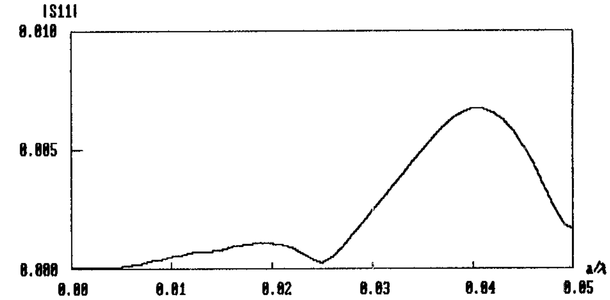

Fig. 5. Reflection coefficient error in FD-TD calculations as a function of the ratio of mesh size to wavelength for a uniform line having a length of 20 meshes.

It should be noted that modeling of a matched load or a matched source is valid only for the TEM mode. In calculations we must assume that the source and the load are placed far enough from the discontinuities to ensure sufficient attenuation of the higher modes. This condition is not difficult to satisfy since all higher modes of axial symmetry have the cutoff frequencies well above the normal range of operation of coaxial lines, and the higher modes are effectively attenuated even at a relatively short distance from the discontinuity. Numerical tests have shown that in most cases a distance equal to the outer conductor radius is sufficient to eliminate error caused by improper matching of the higher modes.

The kth horizontal row of meshes corresponds to coaxial lines of radii $r_k - a/2$ and $r_k + a/2$. The characteristic impedance of such a line is equal to

$$Z'_{0k} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\left(\frac{r_k + a/2}{r_k - a/2}\right) \tag{14}$$

while in the model described by (10)–(12) this impedance corresponds to the admittance

$$Y_{0k} = \frac{1}{2\pi}\sqrt{\frac{C_s}{L_s}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\,\frac{a}{r_k}. \tag{15}$$

In precise calculation of matching conditions for compensated discontinuities, the difference between (14) and (15) can cause additional error. This error can be eliminated by replacing C_s and L_s in (10) and (12) with C'_s and L'_s obtained from integral interpretation of a mesh:

$$C'_s(r_k) = \frac{\mu}{a}\ln\left(\frac{r_k + a/2}{r_k - a/2}\right) \tag{16}$$

$$\frac{1}{L'_s(r_k)} = \frac{1}{\epsilon a}\ln\left(\frac{r_k + a/2}{r_k - a/2}\right). \tag{17}$$

C. Pulse Excitation

As was mentioned above, the calculations are carried out by simulating a pulse excitation of the circuit and calculating the Fourier transform of the signals at the input and the output. If the source voltage as a function of

TABLE I
EIGENFREQUENCIES OF A CYLINDRICAL RESONATOR MODE

Eigenfrequency	TM_{010}	TM_{011}	TM_{020}	TM_{021}	TM_{012}
Calculated analytically	0.3827	0.6297	0.8785	1.0108	1.0707
Calculated numerically	0.3815	0.627	0.872	1.004	1.0645

time $V_0(t)$ is close to the Dirac δ function, the circuit is excited by a very wide frequency spectrum. This allows calculations of wide-band characteristics of the circuit but may have a negative side effect in the case of discontinuity calculations. Exciting the circuit at higher frequencies allows the higher waveguide modes to propagate along the circuit, which prolongs the transient process and also the computing time. To eliminate this effect, the $V_0(t)$ function used in the calculations was chosen to be the δ-type pulse after passing through a bandpass filter having a cutoff frequency ω_c, that is,

$$V_0(t) = \begin{cases} \dfrac{\sin(\omega_c t)}{\omega_c t} & \text{for } |t| < t_1 \\ 0 & \text{for } |t| > t_1. \end{cases} \tag{18}$$

The time limit t_1 was introduced for obvious practical reasons. The value of ω_c is typically assumed to be slightly below the cutoff frequency of the first waveguide mode.

IV. EXAMPLES OF APPLICATION

Some examples of the application of the proposed method will be presented for purposes of comparison with the experimental results or with calculations by other methods. Such relatively simple examples were chosen to test the method because of availability of reference results. It must be stressed that the developed algorithm allows much more complicated discontinuities to be analyzed with no further analytical or programming work.

Example 1: The first example does not exactly concern a discontinuity but its results say quite a lot about the method's accuracy. Let us consider a cylindrical resonator of radius r_0 and length $l = r_0$. Its equivalent planar circuit for TM_{0mn} modes is a square circuit grounded at one side. The circuit is excited by a pulse entering it by an additional line formed of one row of meshes. Minima of the function $J_t(\omega)$ describing the current entering the square circuit indicate the eigenfrequencies of the resonator's TM_{0mn} modes [7]. Table I compares the results of calculations assuming a mesh size $a = r_0/15.5$ with the values obtained analytically. The frequency is normalized such that it is equal to unity for a wavelength equal to r_0. Good accuracy of the eigenvalue calculations for the waveguide modes is a good prognosis for the method's overall accuracy since these modes are generated at discontinuities and determine their properties. The results are also interesting since the circuit considered includes the axis of symmetry, which in the equivalent planar circuit gives $\mu = 0$ and $\epsilon = \infty$. The grid was situated such that the axis was passing

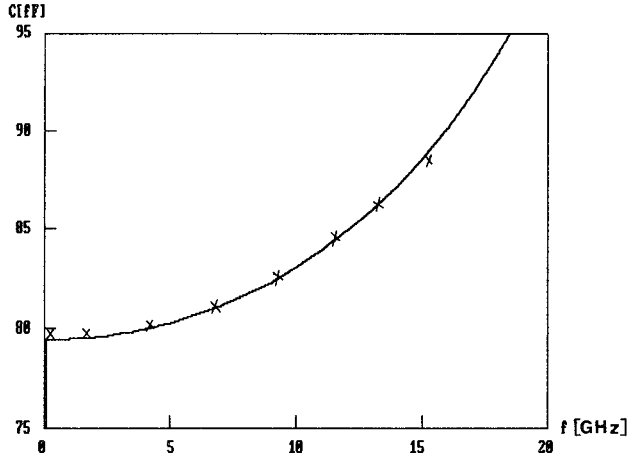

Fig. 6. Capacitance of the open end of a 50 Ω (7 mm) coaxial line terminated in a circular waveguide as calculated by the presented method (continuous line), compared with calculations by Bianco *et al.* [3] (crosses).

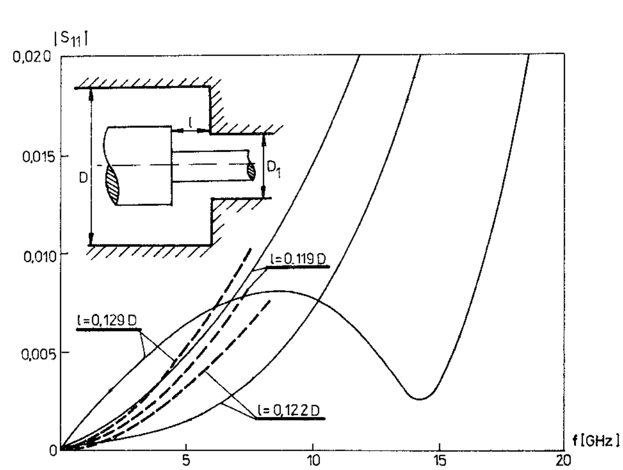

Fig. 7. Reflection coefficient versus frequency for a step transition between two 60 Ω lines with $D = 10$ mm and $D_1 = 3.623$ mm calculated by our method (continuous line) and measured by Kraus [10] (dashed line).

through centers of meshes where $V = 0$ was assumed. No special problems with algorithm stability were encountered.

Example 2: The capacitance of an open-circuited 50 Ω coaxial line terminated in a circular waveguide is considered. The results of calculations are compared in Fig. 6 with those obtained by Bianco *et al.* [3] by a variational method. The agreement is very good. The results of calculations by the FD-TD method were obtained with a mesh size $a = 0.0966D$ (where D is the outer conductor diameter) and practically do not change when the calculations are repeated with a smaller mesh size.

Example 3: A step transition between two 60 Ω lines (with the outer conductor diameters equal to $D = 10$ mm and $D_1 = 3.623$ mm) is considered. In Fig. 7 the results of calculations by our method with a mesh size $a = 0.01636D$ are compared to measurements by Kraus [10].

Example 4: A tapered transition between the lines considered in example 3 is investigated. The taper consists of

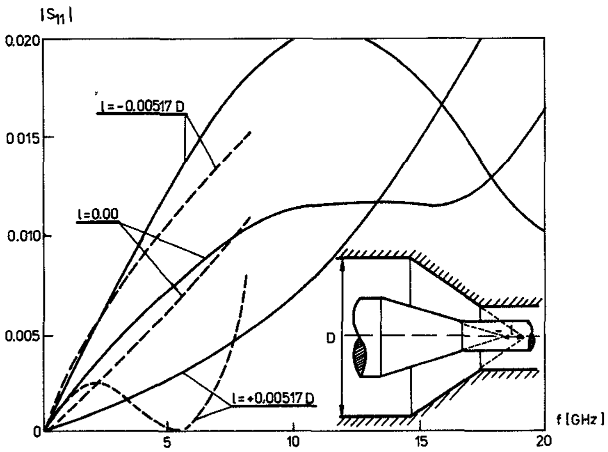

Fig. 8. Reflection coefficient versus frequency for a tapered transition between two 60 Ω lines with $D = 10$ mm and $D_1 = 3.623$ mm calculated by our method (continuous line) and measured by Kraus [10] (dashed line).

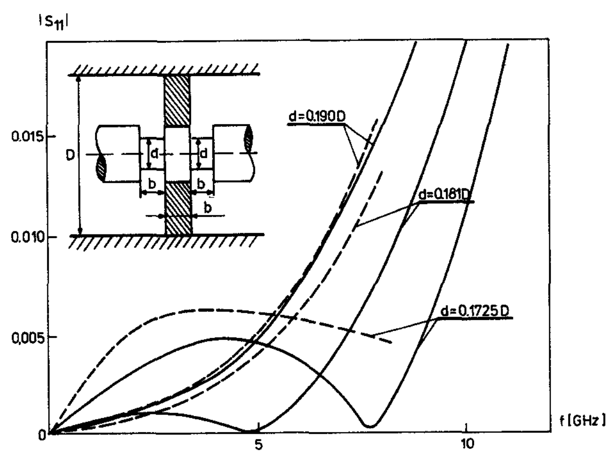

Fig. 9. Reflection coefficient versus frequency for a 60 Ω line with a compensated dielectric support with $D = 10$ mm, $b = 1$ mm, and $\epsilon_r = 2.55$ calculated by our method (continuous line) and measured by Kraus [10] (dashed line).

two cones forming a 60 Ω line. The center of the outer cone is shifted by a distance l left or right with respect to the center of the inner cone. The results of calculations with the same mesh size as in example 3 are compared in Fig. 8 with the measurements by Kraus [10].

Example 5: A compensated support made of a dielectric of $\epsilon_r = 2.55$ is considered. The results of calculations (with mesh size $a = 0.02D$) are compared in Fig. 9, again with the measurements by Kraus [10].

The last three examples concern practical problems for which, to the author's knowledge, no reliable results of calculations have been obtained by any other analytical or numerical method. The only available comparative data are the measurement results by Kraus [10]. The experiments were carried out by Kraus in a very scrupulous way. Line models of large diameters were used. Errors of $|S_{11}|$ measurements were estimated by the author to be smaller than 0.002. The errors in the object dimensions were not estimated. When comparing Kraus's results with our calculations, we have to note that this comparison is done at extremely low levels of the reflection coefficient and takes into account extremely small changes in the circuit's dimensions. Thus it is a very severe test of accuracy for both the measurements and the calculations.

There is very good agreement between our calculations and the measurements by Kraus in the search for the optimum dimensions of the objects, yielding the smallest $|S_{11}|$. There is also fairly good agreement in the character of the curves obtained. In searching for an explanation of the existing differences, we have conducted additional checks on the accuracy of the calculations. Some of them were repeated with increased mesh density and with increased distances between the discontinuity, the source, and the load. No significant changes of the results were observed. This is a strong indication that the calculations were conducted with sufficient accuracy. On the other hand, if we assume the mentioned estimation of the $|S_{11}|$ measurement error and also some reasonable error in establishing dimensions of the objects (for example $0.001D$), experimental error becomes a likely explanation of the differences.

The above examples were calculated on an IBM PC/XT computer with an 8 MHz clock. The time of computing ranged from about 15 min for example 2 to about 80 min for each of the curves of examples 3, 4, and 5. The computing time on a fast mainframe computer should be of the order of several tens of seconds for each of the examples.

V. Conclusions

This paper proposes a method for analyzing arbitrarily shaped coaxial discontinuities. The method produces a universal computer algorithm which permits high-accuracy calculations with no analytical preprocessing. This algorithm can be implemented even on a personal computer, yielding reasonable computing time for solving practical problems. It is shown that the method can be effectively applied to engineering problems.

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